**ONE MISSING OBSERVATION IN GRAECO LATIN SQUARE DESIGN: AN APPROXIMATE ANALYSIS OF VARIANCE**

**Abstract**

In this paper, we proposed an approximate method for one missing observation of Graeco Latin Square of any order. This design adds one more restriction on randomization to the existing missing data for Latin Square by superimposing two orthogonal Latin squares. The existence of missing data for Graeco Latin square is proved and illustrated by considering Latin squares of order 4. A statistical model for the one missing observation was presented and an approximate method for missing data of Graeco Latin Square design was derived and its Mean Square Error was compared with that of complete data in ANOVA procedure. The result of the analysis indicated that the proposed approximate method for Graeco Latin Square is appropriate for estimation of missing observation.

**Keywords:** Latin Square, Graeco Latin Square, ANOVA, missing data, experimental error

**1.0 Introduction**

The result of an experiment can seriously be affected by different degrees of variation that arise from nuisance factors. A nuisance factor can be defined as unknown or uncontrollable design factor that is not of interested to the experimenter which may probably has effect on the response. Sometimes, the existence of this factor is latent and it can even be changing levels while we are performing the experiment. Randomization technique is used to guard such nuisance factors. Blocking is an extremely important design technique that can be used to systematically eliminate the effect of the nuisance factor from statistical comparison among treatments. Graeco-Latin Square design is used to eliminate three sources of variability. That is, it systematically allows blocking in three directions namely: rows, columns and Greek letters that actually represent three restrictions on randomization. Two Latin squares are said to be orthogonal if each pair of letters appears exactly once in the superimposed squares. The superimposed square is called Graeco-Latin square design. There are situations under certain conditions that experimenters might have the problem of a set of incomplete experimental observations. This can be categorized into two situations: (1) the deliberate plan to feature the incomplete observations due to limited number of experimental units. This can be existence of unbalanced arrangement or balance characteristic like Youden square design, the balanced incomplete block design (BIBD), and the balanced incomplete Latin square design (BILSD) proposed respectively by Youden (1937), Yates (1936), and Ai, Li, Liu, and Lin (2013), (2) situations whereby the incomplete observation occurs accidentally which might be as a result of bad control of some variables, the reading values from experiment are abnormal or not observed. Hence, their values might be cut from a set of observations leading to the unbalanced or asymmetrical arrangement.

The analysis of incomplete data problem was first considered by Allan and Wishart (1930), their paper was based on differentiation of overall mean. Yates (1933) and Sirikasemsuk (2016a) proposed non-iterative and iterative missing plot techniques whereby the differential calculus was utilized to determine the missing experimental data with minimal error sum of squares. The estimates of the missing experimental data increases bias of the treatment sum of squares. Thus, the bias is determined and subtracted from the initial treatment sum of squares (Little & Rubin, 2002).

Existing methods to solve the incomplete-data experimental problems are tabulated in Table 1. Many recent studies considered aspects of combinatorics, examples of which were the studies on the construction of the orthogonal Latin squares by Zhang (2013) and Donovan and Şule Yazıcı (2014) and the studies on the completability of the incomplete Latin squares from the partial Latin squares by Euler (2010) and Casselgren & Häggkvist (2013).

Table 1. Existing methods to solve the incomplete-data experimental problems as reported by Kittiwat, S. & Kanogkan, L. (2017)

|  |  |  |  |
| --- | --- | --- | --- |
|  | Method | Description | Author |
| 1 | Missing plot technique by minimizing the error sum of squares with non-iterative method | Differentiating the estimated parameter of the overall mean with respect to each missing value | Allan and Wishart (1930) |
|  |  | General method for estimating several missing values in Latin square design | Kramer and Glass (1960) |
|  |  | Differentiating the error sum of squares to each missing value (when only one observation is missing) | Yates (1933) |
|  |  | Non-iterative Rubin method | Rubin (1972) |
| 2 | Missing plot technique with iterative method Iterative | Yates method (based on the work of Allan and Wishart (1930) when more than one observations are missing) | Yates (1933) |
|  |  | Healy-Westmacott method based on regression imputation  | Healy and Westmacott (1956) |
| 3 | Exact approach with general linear model | General regression significance test | Montgomery (2008) |
| 4 | Multiple imputation (MI) method | A combination of raw maximum likelihood and EM method | Rubin (1987) |
| 5 | Expectation maximization algorithm (EM Algorithm) | Iterative method with maximum likelihood estimation | Dempster, Laird, and Rubin (1977) |
| 6 | Analysis of covariance (ANCOVA) technique | A combination of regression analysis and ANOVA consisting of the covariate | Coons (1957), Cochran (1957), and Wilkinson (1958) |
| 7 | One missing value problem in Latin square design of any order | Exact analysis of variance | Kittiwat S. & Kanogkan L. (2017) |

There are situations whereby one observation is occasionally missed in a Graeco Latin Square design of order P x P, this can pose a serious threat to the analysis of variance if not treated with care. Various methods to handle missing data have been discussed in the literature which include but not limited to multiple imputation method by (Rubin, 1987), Missing plot technique with iterative method by (Yate, 1933), Analysis of covariance (ANCOVA) technique by (Coons, 1957), (Cochran, 1957), and (Wilkinson, 1958) and Expectation maximization algorithm (EM Algorithm) by (Dempster, Laird, and Rubin, 1977). All these methods have successfully estimated missing observation in one way or the other. This work focused at obtaining an approximate missing data method for Graeco Latin Square design.

The rest of the paper is organized as follows. In section 2, we provide the Graeco-Latin Square layout and its Analysis of Variance (ANOVA) table while section 3 discusses derivation of the proposed method and research methodology. Application of the method in ANOVA table are provided in section 4. Discussion of results of the analysis and conclusion are presented in section 5.

2.0 **Graeco-Latin Square design**

The layout of Graeco-Latin Square design is given in table II

Table II: Layout of Graeco-Latin Square design

|  |  |
| --- | --- |
| Row |  Column |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

The null hypothesis of equal row, column, Latin letter and Greek letter treatments would be tested by dividing the corresponding mean square by mean square error. The rejection region is the upper tail point of the distribution.

Table III: Analysis of Variance (ANOVA) for Graeco Latin Square Design

|  |  |  |
| --- | --- | --- |
| Source of variation |  Sum of Square | Degree of freedom |
| Latin letter treatments |  | P-1 |
| Greek letter treatments |  | P-1 |
| Row |  | P-1 |
| Column |  | P-1 |
| Error | SSE(by Subtraction) | (P-3)( P-1) |
| Total |  | P2-1 |

The factor represented by the Greek letters is orthogonal to rows, columns and Latin letters because the Greek letters appear exactly once in each row and column and exactly once with each Latin letter. Therefore a sum of squares due to the Greek letter factor may be computed from the Greek letter totals and the experimental error is further reduced by this amount.

3.0 **Research and Method**

The proposed statistical method and the underlined model is presented in this section

The statistical model for Graeco-Latin square design is

 1

Where

**** grand mean

****ith block one effect (row effect)

**** jth treatment effect

 kth block two effect (column effect)

: lth block three effect (Greek letter)



The Error Sum of Square (SSE) for the model (1) is obtained as equation (2)

SSE=  2

It is possible that an observation in one of the block is missing due to carelessness of error beyond the control of the experimenter such as unavoidable damage to an experimental unit. Due to non-orthogonality of treatment to block, thereby another problem to analyzing the data arises. Hence, we propose an approximate approach to estimate the missing data.

Suppose ‘w’ is missed in Graeco Latin Square then, its SSE is given as obtained in (3)

= 3

Where the primes indicate total for row, column, treatment and Greek letter with missing value while  represent grand total for the missing value and R include all terms not involving w.

Differentiating (3) with respect to w and equate the solution to zero gives us (4)

 4

4.0 **Results and Discussion**

4.1 **Illustrative example of Automobile Emission Experiment**

Data in Table IV is example of an experiment conducted and reported by Zhu (Purdue University 2005) to compare four gasoline additives by testing them on four cars with four drivers over four days. Only four runs can be conducted in each day. The response of the experiment is the amount of automobile emission. The treatment factor is gasoline additive denoted by A, B, C and D, the Block factor1 is driver denoted by 1,2,3,4; Block factor2 is day denoted by 1,2,3,4 and the Block factor 3 is car which is denoted by Greek letters α, β, γ, δ.

|  |
| --- |
|  |
| Driver/Days | 1 | 2 | 3 | 4 |
| 1 | Aα=32 | Bβ=25 | Cγ=31 | Dδ=27 |
| 2 | Bδ=24 | Aγ=36 | Dβ=20 | Cα=25 |
| 3 | Cβ=28 | Dα=30 | Aδ=23 | Bγ=31 |
| 4 | Dγ=34 | Cδ=35 | Bα=29 | Aβ=33 |

 Table IV; Automobile emission experiment

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| Source of Variation | Degree of freedom | Sum of squares | Mean sum of square | F-calculated | F-tabulated |
| Driver | 3 | 310.94 | 103.65 | 1.51 | **3.29** |
| Day | 3 | 68.19 | 22.73 | 0.33 |
| Additive | 3 | 36.69 | 12.23 | 0.18 |
| Car | 3 | 101.19 | 33.73 | 0.49 |
| Error | 3 | 205.94 | 68.65 |  |
| Total | 15 | 722.94 |  |  |

 Table V: Analysis of variance table for Automobile emission experiment

Suppose one observation is completely missed at random, say Dβ=20 in row two column three. Calculation of the missed observation is carried out by equation (4), the estimated observation is obtained to be Dβ=17 by using the proposed method (equation 4). Then, the new ANOVA Table for the missing data is presented in Table VI.

|  |
| --- |
|  |
| Source of Variation | Degree of freedom | Sum of squares | Mean sum of square | F-calculated | F-tabulated |
| Driver | 3 | 108.50 | 36.16 | 4.43 | **3.29** |
| Day | 3 | 89.00 | 29.66 | 3.42 |
| Additive | 3 | 45.50 | 15.16 | 1.86 |
| Car | 3 | 117.50 | 39.16 | 4.80 |
| Error | 3 | 24.50 | 8.16 |  |
| Total | 15 | 385.00 |  |  |

 Table VI: Analysis of variance table for Automobile emission experiment with one missing observation

**Model Adequacy checking**



Figure I: CDF of both missing data experiment and real experiment

**Discussion**

The result of analysis for the proposed method is presented in Table V. Based on this result, Mean Sum of Squares (MSE) reduced drastically compared to that of Table VI for complete data set. Reduction in MSE is a clear indication that with an application of approximate method derived for Graeco-Latin Square Design, we can obtain a better result with a minimum variance unbiased estimate. Despite the sharp reduction of MSE, the conclusion and interpretation of the result is consistent with the result obtained in table VI which is demonstrated in figure1 for model adequacy checking.

**Conclusion**

The research had discussed the existence of one missing observation in Graeco-Latin Square design. Approximate method to address the problem of missing data was derived for the design. The result of the analysis revealed a significant impact of the proposed method which has not altered the conclusion and interpretation of the treatment effect in the analysis. We were able to check for model adequacy of the proposed method by comparing the result with an experiment without missing data through a simulation study of 1000 experimental runs. The result converged with real data set as shown in figure 1. Hence, we recommend that the method derived in this research is capable to handling the problem of missing observation in Graeco-Latin Square design.

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